ISRAEL JOURNAL OF MATHEMATICS 107 (1998), 237–252

# SMOOTH FUNCTIONS ON $C(K)$

BY

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ABSTRACT

We show that every Fréchet differentiable real function on $C(K)$, $K$ scattered with locally uniformly continuous derivative has locally compact derivative. Using this and similar results, we investigate the existence of $C^2$-Fréchet smooth surjections between various Banach spaces.

## 1. Introduction

S. Bates has recently investigated separable Banach spaces $X$ satisfying the condition that for every separable Banach space $Y$ there exists a surjective $C^\infty$-Fréchet smooth (nonlinear) operator from $X$ onto $Y$. We will denote the class of all these spaces by $\mathcal{B}$. Bates has shown that, in particular, every separable superreflexive space belongs to $\mathcal{B}$ and he also characterized spaces for which his method of proof fails.

THEOREM 1 (Bates): *Let $X \notin \mathcal{B}$ be an infinite-dimensional Banach space. Then at least one of the following conditions hold:*

(i) *Every seminormalized weakly null sequence in $X^*$ has a subsequence with a spreading model isomorphic to $\ell_1$.*
(ii) *$X^*$ has the Schur property.*

Natural examples of spaces satisfying (i) or (ii) are $c_0$ and the original Tsirelson space, and Bates asked whether indeed $c_0 \notin \mathcal{B}$. The question was settled in [9]

Received January 23, 1997

(i.e. $c_0 \notin \mathcal{B}$), a paper which was conducted without any knowledge of S. Bates' work, and which was mainly concerned with the behavior of $C^2$-smooth real functions on $c_0$.

In order to reveal the connection between these matters, let us denote by $\mathcal{C}$ the class of Banach spaces $X$ for which there exists an open bounded and convex subset $\mathcal{U}$ of $X$ such that for any real function $f$ defined on $\mathcal{U}$, with uniformly continuous derivative, $f'(\mathcal{U})$ is relatively compact in $X^*$.

Using a sequential characterization of the compactness of $f'(\mathcal{U})$ and the Baire category principle, we show in Proposition 6 that if $X \in \mathcal{C}$, then $X \notin \mathcal{B}$. Thus, whenever $X \in \mathcal{C}$ and $Y \in \mathcal{B}$, there exists no surjective Fréchet differentiable operator $T: X \to Y$ with locally uniformly continuous derivative. A little more can be said under some additional assumptions.

PROPOSITION 2: *Let $X \oplus X \in \mathcal{C}$, $Y$ be an infinite-dimensional Banach space with nontrivial type, $T: X \to Y$ be a Fréchet differentiable operator with locally uniformly continuous derivative. Then $T$ is locally compact.*

The proof of this statement is identical with that of Corollary 11 of [9], using Lemma 5 below instead of Corollary 10 of [9]. It should be noted that some additional assumptions must be put on $Y$ because, as follows from the Josefson–Nissenzweig theorem, every infinite-dimensional Banach space admits a noncompact linear operator into $c_0$.

Proposition 2 is particularly useful if $X \oplus X \cong X$, as is the case when $X = C(K)$, $K$ countable, or $X = T^*$ (the original Tsirelson space) (for these results see [4], [5]). Thus in what follows, we will be mainly interested in showing that $X \in \mathcal{C}$ for these spaces.

In section 2 we collect some basic facts about spaces belonging to $\mathcal{C}$. We develop methods from [9] to show that the original Tsirelson space $T^*$ belongs to $\mathcal{C}$ and, as a consequence, does not admit any surjective operator onto $c_0$ with locally uniformly continuous derivative. Also, $C(K)$, $K$ scattered, belongs to $\mathcal{C}$. On the other hand, the Schreier space $B$ ([5, 11, 12]) yields an example of a polyhedral subspace of $C(\omega^\omega)$ which belongs to $\mathcal{B}$. In particular, $B$ is an example of a subspace of $C(\omega^\omega)$ which is not a quotient of $C(K)$, $K$ scattered.

In section 3 we prove a somewhat finer statement, that there exists no surjective operator from $c_0$ onto $T^*$ with locally uniformly continuous derivative. This suggests that there may be many "incomparable spaces" with respect to smooth surjections.

Section 4 is devoted to proving certain estimates for homogeneous polynomials on $c_0^n$, independent of $n$ and the degree of the polynomial, in the spirit of [2].

We are indebted to R. Haydon who first observed that the methods of [9] apply also in case of the Tsirelson space, and who informed us about S. Bates' work.

Our paper is a natural continuation of [9], but for the convenience of the reader we will repeat some important definitions and statements.

Let $X, Y$ be real Banach spaces. We say that an operator $T: X \to Y$ is locally compact if for every $x \in X$ there exists an open neighbourhood $x \in \mathcal{U}$, such that $T(\mathcal{U})$ is norm relatively compact in $Y$. We say that $T$ is weakly (w)-sequentially continuous on $\mathcal{U} \subset X$ if it maps w-Cauchy sequences from $\mathcal{U}$ into norm convergent ones.

A **modulus of continuity** for a given uniformly continuous function $f$ from a metric space $(X_1, \mathrm{d}_1)$ into a metric space $(X_2, \mathrm{d}_2)$ is an increasing real function $\omega(\delta)$, $\delta \geq 0$, $\lim_{\delta\to 0}\omega(\delta) = 0$, such that

$$\mathrm{d}_1(x_1, x_2) \leq \delta \quad \text{implies} \quad \mathrm{d}_2(f(x_1), f(x_2)) \leq \omega(\delta).$$

The following two statements have been proved in [9], and will be used frequently.

LEMMA 3: *Let $\varepsilon > 0$, $f$ be a real function on $B_{c_0^m}$ with uniformly continuous derivative (with modulus of continuity $\omega(\delta)$) and such that $\sup_{B_{c_0^m}} \|f'\|_1 \leq \omega(2)$. Let $v \in B_{c_0^m}$ and $\{u_i\}_{i=1}^n$ be a block sequence such that $v+u_i \in B_{c_0^m}$. If $n$ is large enough (the estimate depends only on $\omega(\delta)$), then $\min_{1\leq i\leq n} |f(v+u_i)-f(v)| < \varepsilon$.*

LEMMA 4: *Let $f$ be a Fréchet differentiable real function with uniformly continuous derivative defined on $B_{c_0}$. Then $f$ is weakly sequentially continuous on $B_{c_0}$.*

## 2. The class $\mathcal{C}$

In this section we collect some basic facts on spaces from the class $\mathcal{C}$, and we prove that $C(K)$ and $T^*$ (the original Tsirelson space) belong to $\mathcal{C}$.

Let us first remark that if $\ell_1 \hookrightarrow X$, then, by classical results in [7], $\ell_2$ is a linear quotient of $X$, so $X \in \mathcal{B}$. Combined with Proposition 6, this fact yields that $X \in \mathcal{C}$ implies $\ell_1 \not\hookrightarrow X$.

LEMMA 5: *Let $X$ be a Banach space, $\ell_1 \not\hookrightarrow X$. Let $\mathcal{U}$ be an open, bounded and convex subset of $X$ and let $f$ be a real function with uniformly continuous derivative on $\mathcal{U}$. TFAE:*

(i) *$f$ is w-sequentially continuous on $\mathcal{U}$;*
(ii) *$f'(\mathcal{U})$ is relatively compact.*

*If in addition* $x_n \xrightarrow{w} x$, $\{x_n\} \bigcup_{n\in\mathbb{N}}\{x\} \subset \mathcal{U}$, *then* $\lim f(x_n) = f(x)$.

*Proof:* (ii) $\Longrightarrow$ (i). Since $K = \overline{f'(\mathcal{U})}$ is norm compact, given a weakly Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{U}$ we have

$$\lim_{n,m\to\infty} \langle \phi, x_n - x_m\rangle = 0 \quad \text{uniformly in} \phi \in K.$$

By the mean value theorem, for some point $x$ in the interval joining $x_n$ and $x_m$, we have

$$|f(x_n) - f(x_m)| = |\langle f'(x), x_n - x_m\rangle| \le \sup_{\phi\in K} |\langle \phi, x_n - x_m\rangle| \to 0 \quad \text{as } m, n \to \infty.$$

(i) $\Longrightarrow$ (ii). Denote the modulus of continuity of $f'$ on $\mathcal{U}$ by $\omega(\delta)$. Note that $f$ is Lipschitz on $\mathcal{U}$. If $f'(\mathcal{U})$ is not relatively compact, there exist $\varepsilon > 0$ and (by Rosenthal's theorem) a w-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $B(x_n, \varepsilon) \subseteq \mathcal{U}$ and $f_n = f'(x_n)$ satisfy $1/\varepsilon > \|f_n\| > \varepsilon$, $\|f_n - f_m\| > \varepsilon$. If $\lim f(x_n)$ does not exist, we are done. Otherwise, by disregarding quantities that can be made arbitrarily small, we may in addition assume that $f(x_1) = f(x_n), n \in \mathbb{N}$. By induction, we find a subsequence $n_k$ of $\mathbb{N}$ and a sequence $\{y_k\}_{k\in\mathbb{N}}$ in $B_X$ such that

$$(1) \qquad |(f_{n_k} - f_{n_l})(y_l)| > \varepsilon/4 \quad \text{for } k > l,$$

$$(2) \qquad |(f_{n_{k_1}} - f_{n_{k_2}})(y_l)| < \varepsilon/100 \quad \text{for } k_1, k_2 > l.$$

This is done as follows: Choose $y_1 \in B_X$ such that $(f_1 - f_2)(y_1) > \varepsilon/2$. There exists an increasing subsequence $\{n_k^1\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ satisfying (2) for $l = 1$, and satisfying (1) for either $n_1 = 1$ or $n_1 = 2$. Fix the choice of $n_1$ and assume $n_1 < n_1^1$. Find $y_2 \in B_X$ such that $(f_{n_1^1} - f_{n_2^1})(y_2) > \varepsilon/2$. There exists an increasing subsequence $\{n_k^2\}_{k\in\mathbb{N}}$ of $\{n_k^1\}_{k\in\mathbb{N}}$ satisfying (2) for $l = 2$ and (1) for either $n_2 = n_1^1$ or $n_2 = n_2^1$. We continue in an obvious manner.

We may assume that $\{y_k\}_{k\in\mathbb{N}}$ is w-Cauchy. Conditions (1) and (2) imply that for every $k > 3$ we have either $|f_{n_k}(y_{k-2} - y_{k-1})| > \varepsilon/9$ or $|f_{n_{k-1}}(y_{k-2} - y_{k-1})| > \varepsilon/9$. Passing to a suitable subsequence of $\{y_{k-2} - y_{k-1}\}_{k\in\mathbb{N}}$ and $\{f_{n_k}\}_{k\in\mathbb{N}}$, we obtain a w-null sequence $\{z_l\}_{l\in\mathbb{N}}$ such that $f_{n_l}(z_l) > \varepsilon/9$. For $\alpha > 0$ small enough, we have $x_{n_l} + \alpha z_l \in \mathcal{U}$ and $f(x_{n_l} + \alpha z_l) > f(x_{n_l}) + \frac{1}{2}\alpha\frac{\varepsilon}{9}$. This is a contradiction, since $x_1, x_{n_1} + \alpha z_1, x_2, x_{n_2} + \alpha z_2, \dots$ is w-Cauchy.

To prove the last statement, consider the w-Cauchy sequence $\{y_n\} \subset \mathcal{U}$ defined as $y_{2n} = x_n, y_{2n+1} = x, n \in \mathbb{N}$. By (i), there exists a $\lim f(y_n)$, so $\lim f(x_n) = f(x)$. ∎

PROPOSITION 6: *Let $X \in \mathcal{C}$ be a separable Banach space; then $X \notin \mathcal{B}$.*

*Proof:* Let us observe that $\ell_1 \not\hookrightarrow X$. Indeed, otherwise by classical results (see [7]) $\ell_2$ is a linear quotient of $X$. For any convex, bounded open neighbourhood $\mathcal{U}$ of the origin in $X$, there exists a bounded linear operator $T: X \to \ell_2$ such that $B_{\ell_2} \subset T(\mathcal{U})$. The real function $f = \|\cdot\|_2^2 \circ T$ has uniformly continuous derivative on $\mathcal{U}$. The derivative $\mathrm{D}f$ satisfies

$$\mathrm{D}f(x) = \mathrm{D}\|Tx\|_2^2 \circ T = T^*(\mathrm{D}\|Tx\|_2^2).$$

As $2B_{\ell_2^*} = \{\mathrm{D}\|y\|_2^2, y \in B_{\ell_2}\}$ and $T^*$ is noncompact (by Schauder's theorem), $\overline{\mathrm{D}f(\mathcal{U})}$ is noncompact. Consequently, $X \notin \mathcal{C}$; a contradiction. Thus $\ell_1 \not\hookrightarrow X$.

We proceed with the proof by contradiction. Assume $X \in \mathcal{B}$, so there exists a surjective operator $T: X \to \ell_2$ with locally uniformly continuous Fréchet derivative. By shifting and scaling, there exists a covering $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of $X$ consisting of open bounded and convex sets satisfying:

(1) $T\big|_{\mathcal{U}_n}$ has uniformly continuous derivative;
(2) for every real function $f$ defined on $\mathcal{U}_n$ with uniformly continuous derivative, $f'(\mathcal{U}_n) \subset X^*$ is relatively compact.

By the Baire category principle, for some $n \in \mathbb{N}$, $\overline{T(\mathcal{U}_n)}$ contains an open ball $B(x_0, \varepsilon)$, $\varepsilon > 0$.

By Rosenthal's theorem, there exists a w-Cauchy sequence $\{x_k\}_{k\in\mathbb{N}}$ in $\mathcal{U}_n$ such that $\{T(x_k)\}_{k\in\mathbb{N}}$ are $\varepsilon/4$-separated, that is

$$\|T(x_k) - T(x_l)\| > \varepsilon/4 \quad \text{for } k \neq l.$$

Choose a bump function $\phi$ on $\ell_2$ with uniformly continuous derivative and $\phi(0) = 1$, $\phi(x) = 0$ for $\|x\|_2 > \varepsilon/10$. Let $f(x) = \sum_{k=1}^{\infty} \phi(x + x_{2k})$. Then, $f \circ T$ has uniformly continuous derivative on $\mathcal{U}_n$, but $f \circ T$ is not w-sequentially continuous, a contradiction with Lemma 5. ∎

We have the following operator analogue of Lemma 5.

PROPOSITION 7: *Let $X \in \mathcal{C}$ be a Banach space, $T: X \to Y$ be an operator with locally uniformly continuous derivative. TFAE:*

(i) *$T$ is locally w-sequentially continuous;*
(ii) *$T$ is locally compact.*

*Proof:* Find a covering $\{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}$ of $X$ consisting of bounded open and convex sets with the property that

(1) $T\Big|_{\mathcal{U}_\lambda}$ has uniformly continuous derivative;

(2) for every real function $f$ defined on $\mathcal{U}_\lambda$ with uniformly continuous derivative, $f'(\mathcal{U}_\lambda)$ is relatively compact.

We will show that $T\Big|_{\mathcal{U}_\lambda}$ is weakly sequentially continuous if and only if $\overline{T(\mathcal{U}_\lambda)}$ is compact.

(i) $\Longrightarrow$ (ii). $X \in \mathcal{C} \Longrightarrow \ell_1 \not\hookrightarrow X$, so Rosenthal's theorem finishes the proof.

(ii) $\Longrightarrow$ (i). It is enough to use Lemma 5 together with the fact that for every $y^* \in Y^*$, $y^* \circ T\Big|_{\mathcal{U}_\lambda}$ is w-sequentially continuous. ∎

It should be noted that $T$ satisfying the above conditions also locally maps weakly convergent sequences in $X$ into weakly convergent sequences in $Y$. More precisely, let $\mathcal{U}$ be an open bounded and convex set in $X$, such that for every real function on $\mathcal{U}$ with uniformly continuous derivative $\overline{f'(\mathcal{U})}$ is compact, and let $T\colon \mathcal{U} \to Y$ be an operator with uniformly continuous derivative. Then $\{x_n\} \cup \{x\} \subset \mathcal{U}$ and $x_n \xrightarrow{w} x$ implies $T(x_n) \xrightarrow{w} T(x)$. Indeed, by the last part of Lemma 5, $\phi \circ T(x_n) \to \phi \circ T(x)$ for every $\phi \in Y^*$.

PROPOSITION 8: *Let $X \in \mathcal{C}$ be a separable reflexive space, $T\colon X \to Y$ be an onto operator with locally uniformly continuous Fréchet derivative. Then $Y$ is reflexive.*

*Proof:* Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be a covering of $X$ consisting of open bounded and convex sets such that for every real function $f$ defined on $\mathcal{U}_n$, $n \in \mathbb{N}$, with uniformly continuous derivative, $f'(\mathcal{U}_n)$ is relatively compact. By the Baire category principle, for some $\mathcal{U}_n$, $\overline{T(\mathcal{U}_n)}$ has nonempty interior. As $T$ maps weakly convergent sequences from $\mathcal{U}_n$ into weakly convergent sequences in $Y$, $T(\mathcal{U}_n)$ is relatively weakly sequentially compact. By the Eberlein–Šmulyan theorem $Y$ is reflexive. ∎

PROPOSITION 9: *Let $T^*$ be the original Tsirelson space; then $T^* \in \mathcal{C}$.*

*Proof:* Let $f$ be a real function on $B_{T^*}$ with uniformly continuous Fréchet derivative. The space $T^*$ is reflexive and has an unconditional basis $\{e_k\}_{k\in\mathbb{N}}$. By standard argument and using Lemma 5 it is enough to show that $f(x_n)$ converges to $f(0)$ for every w-null sequence in $B_{T^*}$. Assume the contrary, i.e. for some $\{x_n\}_{n\in\mathbb{N}}$, which may be chosen to be a block sequence satisfying $\|x_n\| = l, n \in \mathbb{N}$, and some $\varepsilon > 0$, $|f(x_n) - f(0)| > \varepsilon$. By properties of $T^*$ [5], for every $N \in \mathbb{N}$, $x_N, x_{N+1}, \ldots, x_{2N}$ is $2l$-equivalent to the canonical basis of $c_0^N$. This is a contradiction with Lemma 3. ∎

In particular, there is no $C^2$ operator from $T^*$ onto $c_0$.

Clearly, the same proof also works for $T^*_\theta$, $0 < \theta < 1$ (see [5]), so we have a continuum of mutually totally incomparable reflexive spaces from $\mathcal{C}$. Let us also remark that duals to separable reflexive spaces not belonging to $\mathcal{B}$ cannot contain a copy of $c_0$, $\ell_1$ or a superreflexive space (or even a subspace with nontrivial type).

THEOREM 10: *Let $K$ be a scattered compact. Then $C(K) \in \mathcal{C}$.*

*Proof:* It is enough to show that every real function with uniformly continuous derivative on $B_{C(K)}$ is w-sequentially continuous.

Since every separable subspace of $C(K)$, $K$ scattered, is contained in a separable subspace of $C(K)$ isomorphic to $C(K_1)$, where $K_1$ is countable (e.g. [6]), we may assume that $K$ is countable.

Bessaga and Pelczynski in [4] provided an isomorphic classification of $C(K)$ spaces, $K$ countable, as those isomorphic to $C[0,\alpha]$, where $\alpha$ is a countable ordinal with the interval topology.

We will prove the claim by ordinal transfinite induction on $\alpha$. However, in order to make the induction process work, we are forced to consider a slightly more general setting and introduce auxiliary parameters $F$, $G$, $\alpha_1, \dots, \alpha_r$.

Simple facts on ordinal arithmetics which are used below can be found, e.g., in [10].

We set some notation. Let $\alpha \in \omega_1$. We call a pair $F, G \in C[0,\alpha]$ **admissible** if $\inf_{[0,\alpha]} F > 0$, $\sup_{[0,\alpha]} G < 0$.

Given $\alpha_1, \dots, \alpha_r \leq \alpha$, we define a closed subspace of $C[0,\alpha]$:

$$C_{[0,\alpha]}^{\alpha_1,\dots,\alpha_r} = \{\phi \in C[0,\alpha], \phi(\alpha_i) = 0, 1 \leq i \leq r\}.$$

Given an admissible pair $F, G \in C[0,\alpha]$, we define

$$B^{F,G}_{\alpha,\alpha_1,\dots,\alpha_r} = \{\phi \in C_{[0,\alpha]}^{\alpha_1,\dots,\alpha_r}; G < \phi < F\}.$$

It is easy to verify that for every admissible pair $F$ and $G$, $B^{F,G}_{\alpha,\alpha_1,\dots,\alpha_r}$ is an open convex neighbourhood of the origin in $C_{[0,\alpha]}^{\alpha_1,\dots,\alpha_r}$.

We will prove the following claim by induction on $\alpha$:

*For every admissible pair $F, G$, every $\alpha_1, \dots, \alpha_r \leq \alpha$, and every real function $f$ with uniformly continuous derivative on $\mathcal{U} = B^{F,G}_{\alpha,\alpha_1,\dots,\alpha_r}$, $f$ is w-sequentially Cauchy on $\mathcal{U}$.*

Observe that setting $F \equiv 1$, $G \equiv -1$, $\{\alpha_i\} = \emptyset$ and using Lemma 5 ($\ell_1 \not\hookrightarrow C(K)$, for $K$ scattered) we obtain the desired result.

CASE: $\alpha = \omega_0$. The proof of the special case when $\mathcal{U} = B_{c_0}$ from [9] requires only purely formal adjustments to generalize into case $\mathcal{U} = B^{F,G}_{\omega_0,\alpha_1,\dots,\alpha_{r-1},\omega_0}$, $F, G$ admissible, $\alpha_1, \dots, \alpha_{r-1} \in \omega_0$.

In case none of $\alpha_1, \dots, \alpha_r$ equals $\omega_0$, we proceed by contradiction as follows. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a w-Cauchy sequence in $\mathcal{U} = B^{F,G}_{\omega_0,\alpha_1,\dots,\alpha_r}$, $f$ as above, $f(\phi_n)$ not convergent. By a perturbation argument, we may assume that there exists $C \in \mathbb{R}$, $G(\omega_0) < C < F(\omega_0)$ such that for each $n$, $\phi_n(\alpha) = C$ for $\alpha \in \omega_0$ large enough (depending on $n$). There exists $K \in \omega_0$ such that

$$\inf_{[K,\omega_0]} F > C, \ \sup_{[K,\omega_0]} G < C.$$

Define admissible pair $\tilde{F}, \tilde{G}$ on $C[0, \omega_0]$ as

$$\tilde{F}(\alpha) = \begin{cases} F(\alpha) & \text{for } \alpha \le K, \\ F(\alpha) - C & \text{for } \alpha > K; \end{cases}$$
$$\tilde{G}(\alpha) = \begin{cases} G(\alpha) & \text{for } \alpha \le K, \\ G(\alpha) - C & \text{for } \alpha > K. \end{cases}$$

Consider the affine mapping

$$T\colon C^{\alpha_1,\dots,\alpha_r,\omega_0}_{[0,\omega_0]} \to C^{\alpha_1,\dots,\alpha_r}_{[0,\omega_0]}$$

defined as $T(\phi) = \phi + C \cdot \chi_{[K,\omega_0]}$. The function $f \circ T$ has uniformly continuous derivative on $B^{\tilde{F},\tilde{G}}_{\omega_0,\alpha_1,\dots,\alpha_r,\omega_0}$, but $f \circ T(\phi_n - C \cdot \chi_{[K,\omega_0]})$ is not convergent, a contradiction.

CASE: $\alpha = \omega_0 \cdot \alpha' + \alpha''$, *where* $\alpha', \alpha'' \in \omega_0$. It is enough to repeat the above proofs working simultaneously in all $\alpha'$ copies of $\omega_0$ in the domain.

INDUCTIVE STEP: Assume our claim is true for all $\alpha \in [\omega_0, \beta), \beta \in [\omega_0, \omega_1)$. In addition, let us assume that for no $\gamma < \beta$, $\beta = \gamma + \omega_0$, and $\beta$ is a limit ordinal. Let $\alpha_1, \dots, \alpha_r < \beta$, $F, G$ be an admissible pair from $C[0, \beta]$ and $f$ be a real function with uniformly continuous derivative on $\mathcal{U} = B^{F,G}_{\beta,\alpha_1,\dots,\alpha_r,\beta}$, $f(0) = 0$, $f'(0) = 0$.

Choose an increasing sequence $\beta_k \nearrow \beta$ such that $[\beta_k + 1, \beta_{k+1}]$ are clopen, $\alpha_1, \dots, \alpha_r < \beta_1$. Define for $l < m$, $P^{l,m}\colon C^{\alpha_1,\dots,\alpha_r,\beta}_{[0,\beta]} \to C^{\alpha_1,\dots,\alpha_r,\beta}_{[0,\beta]}$ as

$$P^{l,m}(\phi)(\alpha) = \begin{cases} 0 & \text{if } \alpha \in [\beta_l + 1, \beta_m], \\ \phi(\alpha) & \text{otherwise.} \end{cases}$$

Let us assume, by contradiction, that there is a w-Cauchy sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}$, $f(\phi_{2n}) < 0$, $f(\phi_{2n+1}) > 1$ and each $\phi_n$ is supported by $[0, \beta_i]$ for some $i \in \mathbb{N}$.

Similarly to Claim 7 of [9], we obtain that there is $k \in \mathbb{N}$ and some infinite sets $M_1$ of odd integers and $M_2$ of even integers satisfying, whenever $k \leq l < m$,

$$\begin{aligned} f\big(P^{l,m}(\phi_n)\big) &< \tfrac{1}{4} \quad \text{for all but finitely many } n \in M_2, \\ f\big(P^{l,m}(\phi_n)\big) &> \tfrac{3}{4} \quad \text{for all but finitely many } n \in M_1. \end{aligned}$$

Indeed, otherwise (assuming, e.g., that given any $k$, no such $M_1$ exists) we define a decreasing sequence of infinite subsets$\{M_j\}_{j=1}^{\infty}$ of odd integers and increasing sequences of integers $\{l_i\}_{i=1}^{\infty}, \{m_i\}_{i=1}^{\infty}$, $l_i < m_i < l_{i+1}$, such that $M_{j+1} = \{n \in M_j, f\big(P^{l_i,m_i}(\phi_n)\big) \geq \frac{1}{4}\}$ is infinite. Given any $N \in \mathbb{N}$, we can thus find $\phi_n \in M_N$ for which $|f(\phi_n) - f\big(P^{l_i,m_i}(\phi_n)\big)| \geq \frac{1}{4}$, $i = 1, \ldots, N-1$, a contradiction with Lemma 3.

We may assume, without loss of generality, that $k = 1$, i.e. that the above holds for every $1 \leq l < m$. Using the above claim pass to another subsequence $\{\phi_{p_i}\}_{i \in \mathbb{N}}$, $p_i \in M_2$ for $i$ even, $p_i \in M_1$ for $i$ odd, such that

$$f(\psi_{2j}) < \tfrac{1}{4}, \quad f(\psi_{2j+1}) > \tfrac{3}{4}, \quad \text{where } \psi_i = P^{1,i}(\phi_{p_i}).$$

In addition, we may also assume $\operatorname{supp}(\psi_i) \subset [0, \beta_1] \cup [\beta_i + 1, \beta_{i+1}]$. By construction, $\{\psi_i\}_{i \in \mathbb{N}} \subseteq \mathcal{U}$ is w-Cauchy. Consider the linear operator $L$: $C_{[0,\beta_1+\omega_0]}^{\alpha_1,\ldots,\alpha_r,\beta_1+\omega_0} \to C_{[0,\beta]}^{\alpha_1,\ldots,\alpha_r,\beta}$, defined by

$$L(\phi)(\alpha) = \begin{cases} \phi(\alpha) & \text{if } \alpha \leq \beta_1, \\ \phi(\beta_1 + i) \cdot \psi_i(\alpha) & \text{if } \alpha \in [\beta_i + 1, \beta_{i+1}]. \end{cases}$$

There exists $\eta > 0$ such that for every $i \in \mathbb{N}$

$$G\big|_{[\beta_i+1,\beta_{i+1}]} < -\eta \cdot \psi_i\big|_{[\beta_i+1,\beta_{i+1}]} < F\big|_{[\beta_i+1,\beta_{i+1}]}.$$

Define admissible $\tilde{F}, \tilde{G} \in C[0, \beta_1 + \omega_0]$ as

$$\tilde{F}(\alpha) = \begin{cases} F(\alpha) & \text{for } \alpha \leq \beta_1, \\ 1 & \text{for } \alpha > \beta_1; \end{cases}$$

$$\tilde{G}(\alpha) = \begin{cases} G(\alpha) & \text{for } \alpha \leq \beta_1, \\ -\eta & \text{for } \alpha > \beta_1. \end{cases}$$

The real function $f \circ L$ has uniformly continuous Fréchet derivative on $B_{\beta_1+\omega_0,\alpha_1,\ldots,\alpha_r,\beta_1+\omega_0}^{\tilde{F},\tilde{G}}$. However, $f \circ L\big((\psi_i \cdot \chi_{[0,\beta_1]} + \delta_j^i \cdot \chi_{\{\beta_1+j\}}\big)$ is not convergent, a contradiction with the inductive hypothesis since $\alpha_1 + \omega_0 < \beta$.

The case $\mathcal{U} = B_{\beta,\alpha_1,\ldots,\alpha_r}^{F,G}$, where all $\alpha_i$ are distinct from $\beta$, follows from the previous case in the same fashion as in the case of $c_0$ above, that is to say, an adjustment of the admissible pair $F, G$ allows us to add $\beta$ among the $\alpha_1, \ldots, \alpha_r$.

The only remaining case is when, for some $\gamma \geq \omega^2$, $\beta = \gamma + \delta$ where $\delta \leq \omega_0$. In this case there exists a homeomorphism of the domain $H\colon \gamma + \delta \to \delta + \gamma = \gamma$ defined as

$$H(\alpha) = \begin{cases} \delta + \alpha & \text{if } \alpha \leq \gamma, \\ \alpha' & \text{if } \alpha = \gamma + \alpha'. \end{cases}$$

Thus, we obtain a reduction to the case of $\gamma$, $\gamma < \beta$, and the proof is completed. ∎

The following surprising example, based on a construction of Schreier [12], was investigated in [11].

*Example 11:* There exists a subspace $B$ of $C(\omega^\omega)$ with unconditional shrinking basis $\{e_n\}$ and a biorthogonal basis $\{e_n^*\}$ such that $e_n^* \xrightarrow{w} 0$ and the spreading model built on $\{e_n^*\}$ is $c_0$.

It follows immediately from Theorem 1 that $B \in \mathcal{B}$. Moreover, using [11], one can show by standard argument that the formal canonical injection from $B$ into $\ell_2$ is bounded. Yet, the space $B$ as a subspace of a polyhedral space is itself (isomorphically) polyhedral and thus saturated by copies of $c_0$. The space $B$ also indicates that the structure of w-Cauchy sequences in general $C(K)$, $K$ scattered, is more complicated than that of $c_0$. This is the main obstacle in trying to prove analogous statements to Proposition 13 below for $C(K)$ instead of $c_0$.

## 3. Operators from $\mathbf{c_0}$

The main Proposition 13 of this section implies that a $C^2$-smooth operator from $c_0$ into a space $Y$ with an unconditional basis is locally compact unless $c_0 \hookrightarrow Y$. Together with Proposition 8 this statement implies that there is no surjective $C^2$-smooth operator from $c_0$ onto $T^*$ or vice versa.

LEMMA 12: *Let $T\colon B_{c_0} \to Y$ be an operator with uniformly continuous Fréchet derivative on $B_{c_0}$. Assume that for every given $u \in B_{c_0}$ and $\{v_n\}_{n\in\mathbb{N}} \subset c_0$ equivalent to the canonical basis of $c_0$, such that $u + v_n \in B_{c_0}$, we have*

$$\lim_{n\to\infty} T(u + v_n) = T(u).$$

*Then $T$ is w-sequentially continuous on $B_{c_0}$.*

*Proof:* Assume, by contradiction, that $T$ is not w-sequentially continuous on $B_{c_0}$, i.e. there exists a w-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \in B_{c_0}$ such that $T(x_n)$ is not convergent. If $\{T(x_n)\}_{n\in\mathbb{N}}$ is relatively compact, then there exists $y^* \in Y^*$ such that $\{y^* \circ T(x_n)\}_{n\in\mathbb{N}}$ is not convergent, a contradiction with Lemma 4.

We therefore assume that $\{T(x_n)\}_{n\in\mathbb{N}}$ is not relatively compact. By passing to a subsequence, changing notation and disregarding quantities that can be made arbitrary small, we can assume that there is a w-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \in B_{c_0}$ satisfying:

(i) $\operatorname{dist}\{\operatorname{span}\{T(x_1),\dots,T(x_n)\},T(x_{n+1})\} > \beta > 0$,

(ii) $\{x_n\}$ are supported in an increasing sequence of finite intervals $I_n = [1, m_n]$,

(iii) all the $x_j$, for $j > n$, are equal on $I_n$.

By assumption, for every $x_n$ and every block sequence $\{y_k\}_{k\in\mathbb{N}}$ such that $x_n + y_k \in B_{c_0}$,

$$\lim_{k\to\infty} T(x_n + y_k) = T(x_n).$$

Thus for every $n \in \mathbb{N}$ there exists $l_n \in \mathbb{N}$, $l_n > m_n$ such that $\|T(x_n+u)-T(x_n)\| < \beta/2$ for every $u \in c_0$, $x_n + u \in B_{c_0}$, $\operatorname{supp}(u) \subset [l_n, \infty)$. Consequently, for every $N \in \mathbb{N}$ we can choose a finite sequence $x_{n_1},\dots,x_{n_{2N}}$ satisfying $l_{n_i} < m_{n_{i+1}}, i = 1,\dots,2N-1$. We obtain the following:

$$\|T(x_{n_i} + \chi_{[l_{n_i}, m_{n_{2N}}]} \cdot x_{n_{2N}}) - T(x_{n_i})\| < \beta/2, \quad i = 1,\dots,2N-1.$$

Put $u_{n_i} = x_{n_{2N}} - (x_{n_{2i}} + \chi_{[l_{n_{2i}}, m_{n_{2N}}]} \cdot x_{n_{2N}}), i = 1,\dots,N-1$. Then $u_{n_i}$ is a block sequence satisfying $\|u_{n_i}\| = 2$, supported by $[m_{n_{2i-1}}, l_{n_{2i}}]$. Using (i), choose $y^* \in B_{Y^*}$ satisfying

$$y^*(T(x_{n_i})) = 0, \quad i = 1,\dots,2N-1,$$
$$y^*(T(x_{n_{2N}})) > \beta.$$

Thus we have $|y^* \circ T(x_{n_{2N}} - u_{n_i})| < \beta/2$, $i = 1,\dots,N-1$, $|y^* \circ T(x_{n_{2N}})| > \beta$. Because $N$ is arbitrary large, it is a contradiction with Lemma 3. ∎

PROPOSITION 13: *Let $Y$ be a separable* Banach *space with an unconditional basis. Suppose $T: c_0 \to Y$ has a locally uniformly continuous Fréchet derivative. Then either $c_0 \hookrightarrow Y$ or $T$ is locally compact.*

*Proof:* We proceed by contradiction, assuming that $c_0 \not\hookrightarrow Y$ and $T$ is not locally compact. By standard arguments together with Lemma 12 and Proposition 7, we may assume that $T$ has uniformly continuous derivative on $\{a \in B_{c_0}, a = \sum a_i e_i, -\delta \le a_i \le 1 \text{ for } i \in \mathbb{N}\}$ for some $\delta > 0$, $T(0) = 0$ and $\|T(e_k)\| \ge 2\varepsilon > 0$. Denote by $\{x_k\}_{k\in\mathbb{N}}$ the unconditional normalized basis of $Y$, $\{x_k^*\}_{k\in\mathbb{N}}$ its dual basis. By the proof of Theorem 10 (a variation of Lemma 4), $\{T(e_k)\}_{k\in\mathbb{N}}$ is w-null, so on passing to a subsequence we may assume that there exist a sequence $J_k = [i_k, j_k]$ of consecutive intervals of integers and $f_k \in B_{Y^*}$,

$f_k \in \operatorname{span}\{x^*_{i_k}, \ldots, x^*_{j_k}\}$, such that $f_k \circ T(e_k) > 3\varepsilon/2 > 0$. Put $P^k\colon Y \to Y$ to be a projection defined as $P^k(\sum_{i=1}^{\infty} \alpha_i x_i) = \sum_{i=i_k}^{j_k} \alpha_i x_i$. Our aim now is to pass to a subsequence $\{k_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that:

$$f_{k_l} \circ T\Big(\sum_{i=1}^{n} e_{k_i}\Big) \geq \varepsilon \quad \text{for every } 1 \leq l \leq n.$$

Before we present the construction, let us observe how this implies the statement of Proposition 13. By compactness of bounded sets in finite-dimensional space $P^{k_l}Y$, we may find an increasing sequence of integers $\{n_p\}_{p\in\mathbb{N}}$ such that for every $l \in \mathbb{N}$

$$\lim_{p\to\infty} P^{k_l}\Big(T\Big(\sum_{i=1}^{n_p} e_{k_i}\Big)\Big) = u_l$$

exists, and of course $\|u_l\| \geq \varepsilon$.

By the unconditionality of $\{x_k\}_{k\in\mathbb{N}}$ and boundedness of $T$, $\{u_l\}_{l\in\mathbb{N}}$ forms a block basis in $Y$ satisfying

$$C_1 \max\{|\alpha_l|\} \leq \Big\|\sum_{l=1}^{n} \alpha_l u_l\Big\| \leq C_2 \max\{|\alpha_l|\} \quad \text{where } C_1, C_2 \geq \varepsilon.$$

In other words, $\{u_l\}_{l\in\mathbb{N}}$ is equivalent to the canonical basis of $c_0$. The left inequality follows from the fact that $\{u_l\}$ is a seminormalized block basic sequence. From the unconditionality, we obtain that $\|\sum \alpha_l u_l\| \leq C \max|\alpha_l| \|\sum u_l\|$, and the latter is bounded by the above and by the unconditionality of the basis (which implies that $\sum P^{k_l}$ is bounded). The sequence $\{k_i\}_{i\in\mathbb{N}}$ is constructed by induction as follows. Given $r \in \mathbb{N}$, put $n_r$ to be a large enough integer (Lemma 3) so that whenever $f \in B_{Y^*}$, $v \in B_{c_0}$, $\{w_i\}_{i=1}^{n_r} \in c_0$ are such that $v + w_i \in B_{c_0}$, and $w_i$ are 1-equivalent to the canonical basis of $c_0^{n_r}$, we have

$$|f \circ T(v + w_i) - f \circ T(v)| < \Big(\frac{\varepsilon}{2}\Big)^{r+1}$$

for some $i \in [1, n_r]$.

Using Lemma 3 again, there exists $Q_1 \in \mathbb{N}$, $Q_1 > n_1$ such that $f_i \circ T(e_i + u) \geq (1 + \frac{1}{4})\varepsilon$ whenever $i \in [1, n_1]$, $u \in B_{c_0}, \operatorname{supp}(u) \subset [Q_1, \infty)$. On the other hand, for every $j > Q_1$ there exists some $i \in [1, n_1]$ such that $f_j \circ T(e_i + e_j) \geq (1 + \frac{1}{4})\varepsilon$. Thus there exists $k_1 \in [1, n_1]$ and an infinite increasing sequence $\{m_1^1, m_2^1, \ldots\} = M_1 \subset \mathbb{N}$ such that for every $u \in B_{c_0}, \operatorname{supp}(u) \subset M_1$ and every $k \in M_1$ we have $k > k_1$ and

$$f_{k_1} \circ T(e_{k_1} + u) \geq (1 + \tfrac{1}{4})\varepsilon, \quad f_k \circ T(e_{k_1} + e_k) \geq (1 + \tfrac{1}{4})\varepsilon.$$

Similarly, there exists $Q_2 > m^1_{n_2}$ such that $f_i \circ T(e_{k_1}+e_i+u) \geq (1+\frac{1}{8})\varepsilon$ whenever $i \in \{m^1_1, \ldots, m^1_{n_2}\}$, $u \in B_{c_0}, \operatorname{supp}(u) \subset [Q_2, \infty)$. Also, whenever $j > Q_2$, there exists $i \in \{m^1_1, \ldots, m^1_{n_2}\}$ such that $f_j \circ T(e_{k_1}+e_i+e_j) \geq (1+\frac{1}{8})\varepsilon$. Thus, there exist $k_2 \in \{m^1_1, \ldots, m^1_{n_2}\}$ and an infinite increasing sequence $\{m^2_1, m^2_2, \ldots\} = M_2 \subset M_1$ such that for every $u \in B_{c_0}, \operatorname{supp}(u) \subset M_2$ and every $k \in M_2$ we have $k > k_2$ and

$$f_{k_2} \circ T(e_{k_1} + e_{k_2} + u) \geq (1 + \tfrac{1}{8})\varepsilon, \quad f_k \circ T(e_{k_1} + e_{k_2} + e_k) \geq (1 + \tfrac{1}{8})\varepsilon.$$

The inductive process continues in an obvious manner, at the $r$-th step choosing $k_r \in \{m^{r-1}_1, \ldots, m^{r-1}_{n_r}\} \subset M_{r-1}$ and a subset $M_r \subset M_{r-1}$ satisfying

$$f_{k_r} \circ T\left(\sum_{i=1}^{r} e_{k_i} + u\right) \geq \left(1 + \frac{1}{2^{r+1}}\right)\varepsilon,\ f_k \circ T\left(\sum_{i=1}^{r} e_{k_i} + e_k\right) \geq \left(1 + \frac{1}{2^{r+1}}\right)\varepsilon,$$

whenever $u \in B_{c_0}$, $\operatorname{supp}(u) \subset M_r$ and $k \in M_r$. This finishes the proof. ∎

As an immediate consequence, there exists no $C^2$ operator from $c_0$ onto $T^*$.

## 4. Analytic functions on $\mathbf{c}_0$

In the last part of our paper, we will obtain a finer description of the behavior of real analytic functions on $c_0$, in the spirit of Lemma 4. A similar statement was obtained in the complex setting by Aron and Globevnik in [1]. In fact, using the standard complexification argument, their result implies our Proposition 16.

Our proof uses ideas from [2], but adds a new ingredient of estimating the second derivative, which yields certain estimates independent of the degree of the polynomial and is of independent interest.

We refer to [2] for most of our notation.

Given a real $C^2$-smooth function $f$ on some domain $\mathcal{U}$ in $c^n_0$, we denote by $D^2 f\colon \mathcal{U} \to \mathcal{L}(c^n_0, \ell^n_1)$ the usual second derivative of $f$, which can be represented at every point of $\mathcal{U}$ by a symmetric matrix $(\partial^2 f/\partial x_i \partial x_j)_{i,j=1,\ldots,n}$. For $T \in \mathcal{L}(c^n_0, \ell^n_1)$, $\|T\|$ stands for the usual operator norm. Let us denote $\overline{\Delta} f = \sum_{i=1}^n |\partial^2 f/\partial x_i^2|$. The following lemma is well known; we include the proof for convenience.

LEMMA 14: *Let $T = (a_{ij})_{i,j=1,\ldots,n} \in \mathcal{L}(c^n_0, \ell^n_1)$, $\|T\| = 1$; then $\sum_{i=1}^n |a_{ii}| \leq 1$. In particular, let $f \in C^2$, $f\colon B_{c^n_0} \to \mathbb{R}$, $\|D^2 f\| \leq 1$ on $B_{c^n_0}$. Then $\overline{\Delta} f \leq 1$ on $B_{c^n_0}$.*

*Proof:* Clearly,

$$\|T\| = \max\left\{\|T(x)\|_1, x = \sum_{i=1}^{n} \pm e_i\right\}.$$

For any choices of signs $\varepsilon_j = \pm 1, \delta_i = \pm 1, 1 \le i, j \le n$, we have

$$\|T\| \ge \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} \varepsilon_j a_{ij} \Big| \ge \sum_{i=1}^{n} \Big( \delta_i a_{ii} + \delta_i \sum_{j \ne i} \varepsilon_j a_{ij} \Big).$$

Keeping $\delta_i$ fixed and averaging over all possible combinations of signs of $\varepsilon_j$ we obtain $\|T\| \ge \sum_{i=1}^{n} \delta_i a_{ii}$, so $\|T\| \ge \sum_{i=1}^{n} |a_{ii}|$. ∎

Lemma 15: *Let $p$ be a homogeneous polynomial of degree $k$ on $B_{c_0^n}$. If $\overline{\Delta} p \le 1$ on $B_{c_0^n}$, then $\sum_{i=1}^{n} |p(e_i)| \le 16$.*

*Proof:* We may assume that $n$ is odd and $p$ is a symmetric polynomial, and we need to prove our estimate with 8 rather than 16. Indeed, otherwise assuming $p(e_i) \ge 0$ (here is why we need a better estimate; in general we have to pass to a suitable subset of $\{e_i\}_{i=1}^{n}$, where the signs of $p$ remain constant) we can consider $\tilde{p}$ defined on $B_{c_0^m}$, $m \ge n$, $m$ odd, as

$$\tilde{p}\left(\sum_{i=1}^{m} a_i e_i\right) = \frac{1}{m!} \sum_{\pi \in \Pi_m} p\left(\sum_{i=1}^{n} a_{\pi(i)} e_i\right),$$

where $\Pi_m$ is the group of permutations of $\{1, \dots, m\}$. Clearly, $\tilde{p}$ is symmetric, $\overline{\Delta}\tilde{p} \le 1$ and $\sum_{i=1}^{m} |\tilde{p}(e_i)| = \sum_{i=1}^{n} |p(e_i)|$.

Assume $p(x_1, \dots, x_n) = \sum_{|\alpha|=k} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$; denote $\tilde{a}_i$ the coefficient by $x_i^k$. Clearly, $\sum_{i=1}^{n} |p(e_i)| = \sum_{i=1}^{n} |\tilde{a}_i|$. To estimate $\sum_{i=1}^{n} |\tilde{a}_i|$, consider the polynomial

$$q(x_1, \dots, x_n) = \sum_{i=1}^{n} (-1)^i \frac{\partial^2 p}{\partial x_i^2}.$$

Then $q$ is a homogeneous polynomial of degree $k-2$, $|q| \le 1$ on $B_{c_0^n}$ and, due to the symmetry of $p$ and $n$ being odd, the leading coefficients of $q$ by $x_i^{k-2}$ are $(-1)^i k(k-1)\tilde{a}_i$. By Theorem 1.2 of [2], $k(k-1)\sum_{i=1}^{n} |\tilde{a}_i| \le 4k^2$. Thus $\sum_{i=1}^{n} |\tilde{a}_i| \le 8$, and the proof is completed. ∎

Unfortunately, uniform estimates of this type, independent of the dimension $n$ and degree of the polynomial, are not valid for nonhomogeneous polynomials (consider, e.g., $\prod_{i=1}^{n} (1 - x_i^4)$ on $c_0^n$). This is the reason why no analogue of the following proposition is valid under the weaker assumption of $C^2$ smoothness rather than analyticity.

PROPOSITION 16: *Let $f$ be a real analytic function on some domain $\mathcal{U}$ in $c_0$, $0 \in \mathcal{U}$, $f(0) = 0$ and $f'(0) = 0$. Then there exists some $\varepsilon > 0$ such that $\sum_{i=1}^{\infty} |f(\varepsilon e_i)| < \infty$.*

*Proof:* Let us assume that the Taylor series of $D^2 f$ at 0, namely

$$D^2 f(x) = P_0 + P_1(x) + P_2(x) + \cdots,$$

where $P_k(x)$ is a $k$-homogeneous polynomial form $c_0$ into $\mathcal{L}(c_0, \ell_1)$, is uniformly convergent on $\varepsilon B_{c_0}$ and moreover satisfies

$$\sup_{x \in \varepsilon B_{c_0}} \|P_k(x)\| \leq K(1-\varepsilon)^k,$$

where $K$ is some constant. By Lemmas 14 and 15 and an easy homogeneity argument, we obtain

$$\sum_{i=1}^{\infty} |f(\varepsilon e_i)| \leq 16K\varepsilon^2 \sum_{k=0}^{\infty} (1-\varepsilon)^k = 16K\varepsilon.$$

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